8.5 Partial Fractions (part 2)

last time: linear factors, some may be repeated

$$\frac{1}{(x)(x-2)} = \frac{A}{x} + \frac{B}{x-2} \qquad \text{find } A, B$$

$$\frac{1}{(x)(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \qquad \text{find } A, B, C$$

numerator must have lower degree than denominator

3rd degree

$$\frac{26x^{3} - 52x^2 + 2}{x^{2} - 2x} \qquad \text{must reduce before expansion}$$

2nd degree

last time: rearrangement

this time: long division

$$\frac{26x^3 - 52x^2 + 2}{x^2 - 2x}$$

x^2-2x goes $26x$ times into $26x^3-52x^2+2$

$$\begin{array}{r|l} & 26x \\ \hline x^2-2x & 26x^3 - 52x^2 + 2 \\ & -(26x^3 - 52x^2 \quad) \\ \hline & 0 + \cancel{52x^2} + 2 \\ & \quad\;\; 0 \end{array}$$

remainder

0th degree

$$= 26x + \frac{2}{x^2-2x}$$

2nd degree

$$= 26x + \underbrace{\frac{2}{x(x-2)}}_{\frac{A}{x} + \frac{B}{x-2}}$$

$$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$2 = A(x-2) + B(x)$$

$$\int 26x + \left(\frac{A}{x} + \frac{B}{x-2}\right) dx$$

now quadratic factors

reducible : for example, $x^2 - 2x = (x)(x-2)$ product of linear factors

irreducible : cannot be factored into linear factors

for example, $x^2 + 4$

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{(x)(x^2+4)}$$

linear

irreducible quadratic

linear (1st deg)

for irreducible quadratic

$$= \underbrace{\frac{A}{x}} + \underbrace{\frac{Bx + C}{x^2 + 4}}$$

for the linear factor x

1st degree

numerator is constant (0th degree)

2nd degree

numerator is one degree lower than denominator example

$$\int \frac{x^2+x+2}{(x+1)(x^2+1)}\,dx$$

degree check: numerator(2nd) < denom. (3rd)

ok

expansion: $\dfrac{x^2+x+2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ find A, B, C

linear

irreducible quadratic

multiply by $(x+1)(x^2+1)$

$$x^2+x+2 = A(x^2+1) + (Bx+C)(x+1)$$

$$= Ax^2+A + Bx^2+Bx+Cx+C$$

$$x^2+x+2 = (A+B)x^2 + (B+C)x + (A+C)$$

$A+B=1$ — ①

$B+C=1$ — ②

$A+C=2$ — ③

from ① $B = 1 - A$ sub into ②

②: $B + C = 1$

$1 - A + C = 1$

$C = A$ sub into ③

③: $A + C = 2$

$A + A = 2 \rightarrow \boxed{A = 1}$ $\boxed{C = 1}$ $\boxed{B = 0}$

$$\int \frac{x^2 + x + 2}{(x+1)(x^2+1)}\,dx = \int \left(\frac{A}{x+1} + \frac{Bx + C}{x^2+1} \right) dx$$

$$= \int \left(\frac{1}{x+1} + \frac{1}{x^2+1} \right) dx = \int \frac{1}{x+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

trig subs

$$= \boxed{\ln|x+1| + \tan^{-1}(x) + D}$$

repeated irreducible quadratic factors are handled just like how we handle repeated linear factors

for example, $\dfrac{1}{(x)(x^2+1)^2} = \dfrac{1}{\underbrace{(x)}_{\text{linear}}\underbrace{(x^2+1)(x^2+1)}_{\text{irreducible and repeated}}}$

$$\frac{1}{(x)(x^2+1)(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$ find A, B, C, D, E

multiply by $(x)(x^2+1)(x^2+1)$

$$1 = A(x^2+1)(x^2+1) + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

⋮ multiply out, collect by power

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$0x^4 + 0x^3 + 0x^2 + 0x + 1$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

right away we see $\boxed{C = 0}$, $\boxed{A = 1}$

then $A + B = 0 \rightarrow B = -A \rightarrow \boxed{B = -1}$

then $C + E = 0 \rightarrow E = -C \rightarrow \boxed{E = 0}$

$2A + B + D = 0$

$2(1) + (-1) + D = 0 \rightarrow \boxed{D = -1}$

$$\int \frac{1}{(x)(x^2+1)(x^2+1)}\,dx$$

$$= \int \left(\frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2} \right) dx$$

$$= \int \left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right) dx = \underbrace{\int \frac{1}{x}\,dx}_{\text{easy}} - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

sub $u = x^2 + 1$ →

$\vdots$

$$= \boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\,\frac{1}{x^2+1} + F}$$

write the form of expansion

$$\frac{2x^3+5x-10}{x^2(x^2+4)^3(x^2-9)^2} = \frac{2x^3+5x-10}{(x)(x)(x^2+4)(x^2+4)(x^2+4)(x+3)(x+3)(x-3)(x-3)}$$

$(x+3)^2(x-3)^2$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2} + \frac{Gx+H}{(x^2+4)^3} + \frac{I}{x+3} + \frac{J}{(x+3)^2} + \frac{K}{x-3} + \frac{L}{(x-3)^2}$$